Final exam "Optimization B". December, 11, 2018

Teacher : Philippe Bich. Delay : 16H30-18H30, 2H, no documents. **Every exit is definitive.** A bad presentation can be penalized of 5% of the total of the grade.

There are 4 exercises. Each exercise represent approximately the same amount of points.

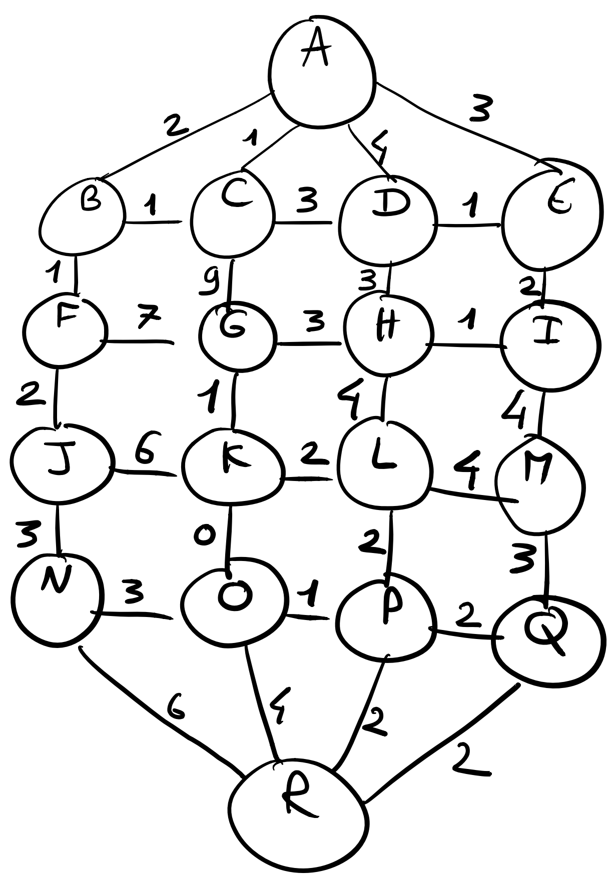
All answers or affirmations have to be fully justified, i.e. if you use theorem, quote it, if you use precise properties, quote them and prove they are true, etc.

Exercise 1

On the following map, you have cities (the nodes), and roads (edges between nodes ; an edge is the "interval" between two cities). Each number on the road between two cities gives the distance on this road between these 2 cities. Two cities being given, you have several paths between these cities : for example, to go from A to E, you can take the direct path (3 kilometers) or go through D then E (4+1=5 kilometers). You want to go from A (departure) to R (arrival). **Remark that all roads between cities have double directions**.

For every city (Z may be A or B or C, ...), we call $V(Z)$ the distance of the shortest path from Z to R. Compute $V(Z)$ for every point $Z = A, B, ..., R$. Find one shortest path from A to R, and its length.

(**Remark : Please** you can use the sheet given with the graphics, and filled each nodes Z labeled A, B, ...,R with the corresponding value of $V(Z)$, and then you can show explicitely the shortest path from A to R with arrows. Please also explicit your method to compute $V(A)$ (for the other points, you can simply put the values). The explanation of your algorithm should be clear and well written, and will be more important than the result.



Exercise 2

Consider the following problem : the initial state at date $t = 1$ (initial date) is $k_1 > 0$. It corresponds to an initial capital. You have to choose your action (which corresponds to a consumption) c_1 and c_2 at each date ($t = 1, t = 2$). At the last date $t = 3$, there is no consumption possible (i.e. $c_3 = 0$). The states k_2 and k_3 corresponds to the remaining capital at time $t = 2$ and $t = 3$.

We assume that $k_{t+1} = k_t - c_t$ for every $t = 1, 2$. The set of possible states (here capitals) is $S = [0, +\infty[$ and the set of possible actions (here consumptions) is $A = [0, +\infty[$.

Your instantaneous utility at time $t = 1, 2, 3$ is assumed to be $U(k_t, c_t) = c_t^{\frac{1}{3}}$, and we assume that the discount factor is $\beta \in]0, 1[$.

1) **General problem and Existence of an optimal consumption plan c_1^*, c_2^*.**

Write mathematically the maximization problem of the intertemporal utility (i.e. the sum of instantaneous utilities) from $t = 1$ to $t = 3$, with respect to the consumptions. call it (P). Prove the existence (without solving it) of an optimal consumption plan c_1^*, c_2^* of (P).

2) **Backward induction method.**

a) If c_1^*, c_2^* is an optimal consumption plan, prove mathematically that $c_2^* = k_2$

b) By Backward induction, deduce that c_1^* is the solution of a one variable maximization problem. Solve it and find c_1^* as a function of k_1 and β. Find also c_2^* as a function of k_1 and β.

c) If the interest rate β converges to zero, what is the limit of the optimal consumption c_1^* at period 1 ?

d) Prove by an explicit computation that if the initial capital k_1 increases, then the optimal consumptions c_1^* and c_2^* both increases.

e) Find by an explicit computation the effect on the optimal consumptions c_1^* and c_2^* of an increase of the interest rate β.

Exercise 3

We recall **Blackwell's Theorem** : *Let $\mathcal{B}(X)$ (endowed with $\|.\|_\infty$) the set of bounded functions from some nonempty subset X to* $\mathbf{R}$. *Let L some closed (for the norm $\|.\|_\infty$) vector subspace of $\mathcal{B}(X)$ containing every constant function. Let $T : L \to L$ such that the following properties i) and ii) are true :*

i) T is increasing, in the following sense : if $f(x) \leq g(x)$ for every $x \in X$, then $T(f)(x) \leq T(g)(x)$ for every $x \in X$

ii) There exists $\beta \in]0, 1[$ such that for every constant function c and every $f \in L$, we have $T(f+c)(x) \leq T(f)(x) + \beta.c$ for every $x \in X$.

Then T admits a fixed point.

1) Recall the definition of "L closed for the norm $\|.\|_\infty$".

2) Recall why $\|f\|_\infty = \sup_{x \in X} \mid f(x) \mid$ defines a norm on $\mathcal{B}(X)$.

3) Recall the proof of Blackwell Theorem.

4) Prove explicitely that the fixed-point of T is unique.

Exercise 4

We consider the maximization problem with respect to feasible actions a_n ($n \geq 0$)

$$\max \sum_{n=0}^{+\infty} \beta^n U(s_n, a_n)$$

wherer a_n is an action you have to choose at period $n \geq 0$, s_n the state at period $n \geq 0$ (for example a capital), $u(s_n, a_n)$ the instantaneous utility at period $n \geq 0$ if state is s_n and if the action chosen at period n is a_n, and β a discount factor.

We assume $U(s_n, a_n) = -a_n$, and we assume that the state at period $n+1$ is defined through the state at period n and the action at period n as follows : $s_{n+1} = 2s_n - a_n$.

Last, we assume that the set of possible states is $S = [0, +\infty[$ and the set of possible actions is $A = [0, +\infty[$.

1) Write Bellman equation satisfied by the optimal value function $V(s)$.

2) Prove that the two functions $V_1(s) = 0$ and $V_2(s) = -2s$ are solutions of Bellman equations.

3) Prove that only one of these two functions is the good optimal value of our problem.