Probability and Statistics
Final exam
No document – No calculator – 2 hours

Notations

The cumulative distribution function of a standard normal (i.e. $\mathcal{N}(0,1)$) random variable is denoted by Φ. We recall that

$$\forall x \in \mathbb{R}, \qquad \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx.$$

We also recall the classical quantiles:

$$\Phi^{-1}(0.9) \simeq 1.28, \quad \Phi^{-1}(0.925) \simeq 1.44, \quad \Phi^{-1}(0.95) \simeq 1.64, \quad \Phi^{-1}(0.975) \simeq 1.96.$$

Basic questions *(5 points)*

1. *(1.5 points)* A couple has $n \geq 2$ children. One of them is a girl. What is the probability that the couple has at least one son?
2. *(1.5 points)* 4 people independently choose randomly (with a uniform distribution) a number in $\{1, 2, 3\}$. Find the probability that at least one person chooses the number 2.
3. *(2 points)* 3 people (A, B and C) play a game with a fair die. A rolls the die, then B, then C, then A, then B, and so on. The game stops when a 6 appears and the winner is the first player who obtained 6. What is the respective probability for A, B, and C to be the winner.

Continuous variables *(3 points)*

Let X be a standard normal random variable.

1. *(1 point)* Show that $\forall k \in \mathbb{N}, \quad \mathbb{E}[X^{2k+1}] = 0$.
2. *(2 points)* Show that $\forall k \in \mathbb{N}, \quad \mathbb{E}[X^{2k}] = \frac{(2k)!}{2^k k!}$.

At the grocery store *(12 points)*

Part I (about suppliers)

A melon is classified as "extra" when its weight is between 900g and 1.2kg.

A grocery store buys melons to three (3) different suppliers denoted respectively by A, B, and C:

- The weight (in gram) of a melon coming from supplier A is modeled by a random variable M_A following a uniform distribution over $[850, x]$, where x is a real number satisfying $x \geq 1200$.
- The weight (in gram) of a melon coming from supplier B is modeled by a random variable M_B following a Gaussian distribution with mean 1050 and standard deviation σ, where σ is a positive real number.

- Supplier C claims that 80% of its melons are "extra".

1. *(1 point)* The manager of the grocery store figures out that 75% of the melons coming from supplier A are "extra". Find x.
2. *(2 points)* The manager of the grocery store figures out that 85% of the melons coming from supplier B are "extra". Determine σ and give a numerical approximation.
3. *(2.5 points)* Over a sample of 400 melons coming from supplier C, the manager of the grocery store noticed that 294 are "extra". Should he believe the claim of supplier C? To answer the question, you must use the central limit theorem and build a statistical test with a confidence level of 95%.

Part II (about clients)

The manager of the grocery store noticed two facts about clients:

- Among clients who buy a melon a given week, 90% buy a melon the following week.
- Among clients who do not buy a melon a given week, 60% do not buy a melon the following week.

Let us choose a client uniformly among those who bought a melon during week 1.

Let us define for $n \in \mathbb{N}^*$ the event A_n : "The client bought a melon during week n". We denote by A_n^c the complementary of that event. For $n \in \mathbb{N}^*$, we denote by p_n the probability $\mathbb{P}(A_n)$. In particular $p_1 = 1$.

1. *(1.5 points)* Compute $\mathbb{P}(A_2)$, $\mathbb{P}(A_2^c)$, $\mathbb{P}(A_3|A_2)$, $\mathbb{P}(A_3|A_2^c)$, $\mathbb{P}(A_3^c|A_2)$, and $\mathbb{P}(A_3^c|A_2^c)$.
2. *(1 point)* Show that $p_3 = 0.85$.
3. *(1 point)* Given that the client bought a melon during week 3, what is the probability that he also bought a melon during week 2?
4. *(1 point)* Show that $\forall n \in \mathbb{N}^*, \quad p_{n+1} = 0.5p_n + 0.4$.
5. *(1 point)* Show that $\forall n \in \mathbb{N}^*, p_n > 0.8$. Deduce that $(p_n)_n$ is decreasing. Is it convergent? If so, what is the limit?
6. *(1 point)* Give p_n in closed form.